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PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS: A SURVEY, (U)  
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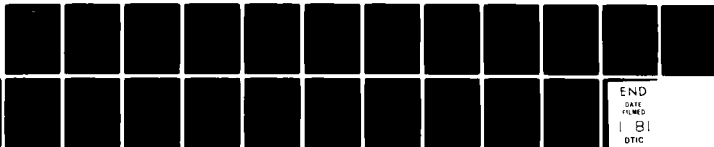
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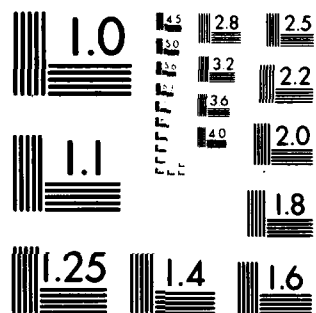
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Periodic solutions of Hamiltonian systems: a survey,

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To Michael Golomb

Periodic solutions of Hamiltonian systems: a survey

Hamiltonian systems of ordinary differential equations model the motion of a discrete mechanical system. During the past few years there has been a considerable amount of progress in the study of periodic solutions of such systems with many new ideas and methods of solution being introduced. The purpose of this paper is to survey these recent developments and their connection with some earlier results. In particular the main results that have been obtained will be stated and an indication will be given of their proofs. A few open questions will also be mentioned. → 19

Let  $p, q \in \mathbb{R}^n$  and  $H: \mathbb{R}^{2n} \rightarrow \mathbb{R}$  be differentiable. An autonomous Hamiltonian system has the form:

$$(0.1) \quad \dot{p} = -\frac{\partial H}{\partial q}(p, q), \quad \dot{q} = \frac{\partial H}{\partial p}(p, q)$$

where  $\cdot$  denotes  $\frac{d}{dt}$ . This system can be represented more concisely as

$$(HS) \quad \dot{z} = J H_z(z)$$

where  $z = (p, q)$  and  $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ ,  $I$  being the identity matrix in  $\mathbb{R}^n$ . Also of interest is the forced analogue of (HS):

$$(FHS) \quad \dot{z} = J H_z(t, z)$$

where  $H$  depends explicitly on  $t$  in a time periodic fashion.

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There are many types of questions, both local and global, that have been studied for (HS) and (FHS). One set of questions has been motivated by the fact that  $H$  is an integral of the motion for (HS), i.e. if  $z(t)$  satisfies (HS),  $H(z(t))$  is independent of  $t$ . Thus one can seek solutions of (HS) having prescribed energy and ask what geometrical properties must an energy surface possess in order for there to exist periodic orbits of (HS) on it. Multiplicity questions are also natural: how many geometrically distinct periodic solutions can there be on a given energy level. Other questions of interest are the existence of solutions of (HS) having a prescribed period and of (FHS) having the given period of forcing. In the setting of (FHS) one can also study the existence of subharmonic solutions, i.e. solutions having a period which is an integer multiple of the period of forcing. Concerning local questions, perturbations of equilibrium or periodic solutions lead to problems of continuation and bifurcation.

The underlying theme in the recent treatment of these problems has been the use of the calculus of variations in finding solutions as critical points of a functional. There have been approaches to (HS) and (FHS) from three main directions: (i) differential geometry—obtaining solutions as geodesics in an appropriate metric; (ii) the direct methods of the calculus of variations—obtaining solutions by minimax arguments from indefinite functionals; (iii) convex analysis and optimization theory—obtaining solutions for convex  $H$  using tools such as the Legendre transformation to simplify the problem.

We will mainly concern ourselves with the existence of periodic solutions of (HS) and (FHS) in the large. However to add perspective some local results also will be mentioned. This will be done in §1. Global results will be described in §2.

## §1. Local results

This section is concerned with some local results for (HS) and (FHS). The early work in this direction had an analytical flavor while the more recent research makes essential use of topological arguments.

We begin with a study of (HS). To normalize matters, let  $H(0) = 0$ . We further assume  $H_z(0) = 0$  so  $z \equiv 0$  is a solution of (HS). The question of interest then is the existence of time periodic solutions of (HS) which are small in amplitude. An old result of Lyapunov [1]—the Lyapunov Center Theorem—applies to this situation:

Theorem 1.1: Suppose  $H$  is twice continuously differentiable near 0,  $H_z(0) = 0$ , and the spectrum of  $\mathcal{J}H_{zz}(0)$ ,  $\sigma(\mathcal{J}H_{zz}(0)) = \{\pm i\zeta_1, \dots, \pm i\zeta_m\}$  where  $\zeta_j$  is real,  $1 \leq j \leq n$ . If  $\zeta_j/\zeta_1$  is not an integer for  $j \neq 1$ , then (HS) possesses a one parameter family of periodic solutions  $z_s(t)$  whose periods  $T(s) \rightarrow 2\pi/\zeta_1$  as  $s \rightarrow 0$ .

Actually Lyapunov looked at a more general situation than (HS). After some simplifications, the proof of Theorem 1.1 can be reduced to the implicit function theorem. If further "nonresonance" assumptions are made on the numbers  $\zeta_j$ , (HS) possesses  $n$  distinct one parameter families of solutions near  $z = 0$ . Thus if  $H(z) > 0$  for small  $z \neq 0$ , these curves of solutions will pierce  $H^{-1}(c)$  for small  $c > 0$  and  $H^{-1}(c)$  contains  $n$  geometrically distinct periodic solutions of (HS). Many attempts were made to obtain similar results without having to impose nonresonance or irrationality assumptions on  $\{\zeta_j\}$ . See e.g. Gordon [2] for such a partial result. No major successes were achieved however until 1973 when A. Weinstein [3] proved:

Theorem 1.2: Suppose  $H$  is twice continuously differentiable near  $0$ ,  $H_z(0) = 0$ , and  $H_{zz}(0)$  is positive definite. Then for all sufficiently small  $c > 0$ , (HS) possesses  $n$  geometrically distinct periodic solutions on  $H^{-1}(c)$ .

Other versions of the result permit the assumption on  $H_{zz}(0)$  to be weakened somewhat [3, 4]. Weinstein's original proof of Theorem 1.2 relies on tools from the theory of Lagrangian manifolds. Moser [4] presents a simpler proof using a variant of the method of Lyapunov-Schmidt to reduce the problem to that of finding critical points of a  $C^1$  function on  $S^{2n-1}$ , the function being invariant under a fixed point free  $S^1$  action. A standard minimax theorem then provides  $n$  geometrically distinct critical points. Other results on perturbation of periodic solutions can be found in Bottkol [5] and Weinstein [6].

It is interesting to note that Theorem 1.2 can be interpreted as the  $S^1$  version, in its setting, of a bifurcation theorem involving functionals with a  $\mathbb{Z}^2$  symmetry due to Bohme [7] and Marino [8]. They proved that if  $E$  is a real Hilbert space and  $f \in C^2(E, \mathbb{R})$  with  $f$  even,  $f'(u) = Lu + H(u)$ ,  $L$  being linear and  $H(u) = o(\|u\|)$  at  $u = 0$ , then if  $\mu \in \sigma(L)$  is an isolated eigenvalue of multiplicity  $n$ , the equation  $f'(u) = \lambda u$  has, for each sufficiently small  $r > 0$ , at least  $n$  distinct pairs of solutions  $(\lambda, \pm u)$  with  $\|u\| = r$  near  $(\mu, 0)$ . (Here  $f'(u)$  denotes the Frechet derivative of  $f$ . Using the duality between  $E$  and  $E'$ , it can be interpreted as a mapping of  $E$  to  $E'$ ). The work of Böhme and Marino motivated Fadell and the author to study the existence of solutions to  $f'(u) = \lambda u$  as a function of  $\lambda$  for  $\lambda$  near  $\mu$  [9]. Applying these ideas to (HS) where  $S^1$  symmetries occur in a natural fashion when seeking periodic

solutions, we considered solutions of (HS) near a bifurcation point as a function of the period [10] and showed:

Theorem 1.3: Suppose  $H$  is twice continuously differentiable near 0 and  $H_z(0) = 0$ . Let  $R^{2n} = E_1 \oplus E_2$  where  $E_1$  and  $E_2$  are invariant subspaces for the flow given by

$$(1.4) \quad \dot{w} = JH_{zz}(0)w.$$

Suppose all solutions of (1.4) with initial data in  $E_1$  are  $T$  periodic, no solutions of (1.4) with initial data in  $E_2 \setminus \{0\}$  are  $T$  periodic, and there are no equilibrium solutions of (1.4) in  $E_1 \setminus \{0\}$ . If the signature  $2\nu$  of the quadratic form  $(H_{zz}(0)\zeta, \zeta)$ ,  $\zeta \in E_1$  is nonsingular, then either (i) every neighborhood of  $z = 0$  contains  $T$  periodic solutions of (HS) or (ii) there are a pair of integers  $k, m \geq 0$  such that  $k + m \geq |\nu|$  and left and right neighborhoods  $J_l, J_r$  of  $T$  in  $R$  such that for all  $\lambda \in J_l$  (resp.  $J_r$ ), (HS) possesses at least  $k$  (resp.  $m$ ) distinct nontrivial  $\lambda$  periodic solutions.

The proof of Theorem 1.3 is related to that of Theorem 1.2 sketched above. To begin one seeks solutions of (HS) in an infinite dimensional space of periodic functions. The method of Lyapunov-Schmidt reduces the problem to a finite dimensional one and a minimax argument relying on an  $S^1$  symmetry inherent in the problem gives the solutions as critical points of a variational formulation of (HS). The minimax construction of the critical points here is more subtle than in [4] due to the fact that there is no analogue of the energy surface constraint of Theorem 1.2 here so one is working in a neighborhood of 0 rather than on a compact manifold. A special case of Theorem 1.3 was obtained independently by Chow and Mallet-Paret [11].



We conclude this section by stating a local theorem concerning the existence of subharmonic solutions of (FHS) due to Birkhoff and Lewis [12-14]. Some preliminary remarks are necessary. If  $H$  is smooth near  $z = 0$ ,  $H(t, 0) = 0$  and  $H_z(t, 0) = 0$ , then  $H(t, z) = Q(t, z) + R(t, z)$  where  $Q$  is quadratic in  $z$  and  $R(t, z) = o(|z|^2)$  at  $z = 0$ . If the Floquet exponents for the linear Hamiltonian system corresponding to  $Q$  are purely imaginary, Floquet theory and the Hamiltonian character of (FHS) permit canonical changes of dependent variables so that the transformed problem has a time independent quadratic part of the form

$$(1.5) \quad \sum_{i=1}^n \lambda_i \frac{p_i^2 + q_i^2}{2}.$$

(See Arnold [15]). Thus we can assume  $Q = Q(z)$  and has the form (1.5). Let  $\Lambda = (\lambda_1, \dots, \lambda_n)$  and let  $k$  be a multiindex,  $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ . Set  $|k| = k_1 + \dots + k_n$  and  $\langle \Lambda, k \rangle = \sum_{i=1}^n \lambda_i k_i$ . If  $H$  is  $C^4$  with respect to  $z$  near 0 and  $\langle \Lambda, k \rangle \notin \mathbb{Z}$  for all  $|k| \leq 4$ , then there exists a canonical change of variables which transforms  $H$  into the Birkhoff normal form:

$$(1.6) \quad Q(z) + \sum_{i,j=1}^n \alpha_{ij} \left( \frac{p_i^2 + q_i^2}{2} \right) \left( \frac{p_j^2 + q_j^2}{2} \right) + o(|z|^4)$$

at  $z = 0$ . (See [15] or Siegel-Moser [16]). Now we can state:

**Theorem 1.7:** Suppose  $H$  is  $C^4$  near 0 and periodic in  $t$  with  $H(t, z) = Q(z) + R(t, z)$ ,  $Q$  as in (1.5) and  $R(t, z) = o(|z|^2)$  at  $z = 0$ . If  $\langle \Lambda, k \rangle \notin \mathbb{Z}$  for all  $|k| \leq 4$  and  $\det(\alpha_{ij}) \neq 0$  in (1.6), then (FHS) possesses a sequence of subharmonic solutions  $(Z_k)$  with arbitrarily large minimal periods.

Above  $\det (\alpha_{ij})$  denotes the determinant of the matrix  $(\alpha_{ij})$ .

There is also a variant of Theorem 1.7 for (HS) [17]. A proof of Theorem 1.7 but in the setting of maps can be found in Moser [18]. A less general version of Theorem 1.7 in the above setting is proved in Harris [17].

## § 2. Global results

In this section the existence in the large of periodic solutions of (HS) and (FHS) will be studied. Our presentation will be roughly chronological. The first work we know of in a global setting is due to Seifert [19] who considered Hamiltonians consisting of the sum of a kinetic and a potential energy term. He essentially proved:

Theorem 2.1: Let  $H(p, q) = \sum_{i,j=1}^n a_{ij}(q) p_i p_j + V(q)$  where  $a_{ij}, V \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ , the matrix  $(a_{ij}(q))$  is uniformly positive definite in  $\mathcal{D} = \{q \in \mathbb{R}^n \mid V(q) < 1\}$  and  $V$  satisfies

(V<sub>1</sub>)  $\mathcal{D}$  is  $C^2$  diffeomorphic to the unit ball in  $\mathbb{R}^n$ ,

(V<sub>2</sub>)  $\partial\mathcal{D}$  is a manifold.

Then there exist points  $q^*, Q^* \in \partial\mathcal{D}$ ,  $T > 0$ , and a solution  $(p(t), q(t))$  of (HS) such that  $(p(0), q(0)) = (0, q^*)$ ,  $(p(T), q(T)) = (0, Q^*)$ , and  $q(t) \in \mathcal{D}$  for  $0 < t < T$ .

Thus Theorem 2.1 gives us a solution whose motion begins and ends on the boundary of the potential well  $\mathcal{D}$ . Seifert actually assumed real analyticity for  $a_{ij}$  and  $V$  but  $C^2$  suffices for his arguments. Observing that  $H$  is even in  $p$ , a  $2T$  periodic solution of (HS) on  $H^{-1}(1)$  can be constructed by extending  $q$  as an even function and  $p$  as an odd function about 0 and  $T$ .

The existence of periodic solutions having a prescribed mean potential energy was studied by Berger [20] for a class of second order Hamiltonian systems which have a less general potential energy term than given above.

Theorem 2.1 was generalized by A. Weinstein [21] who permitted a wider class of kinetic energy terms:

Theorem 2.2: Suppose  $H(p, q) = K(p, q) + V(q)$  where  $K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ ,  $V \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ ,  $V$  satisfies  $(V_1)$ ,  $(V_2)$ , and  $K$  satisfies

$(K_1)$   $K$  is even and strictly convex in  $p$  for each  $q \in \bar{D}$ .

$(K_2)$   $K(0, q) = 0$  and  $K(p, q) \rightarrow \infty$  as  $|p| \rightarrow \infty$  uniformly for  $q \in \bar{D}$ .

Then the conclusions of Theorem 2.1 obtain.

Seifert used ideas from differential geometry to prove Theorem 2.1. Roughly speaking he found the solution as a geodesic for a Riemannian metric (called the Jacobi metric) associated with the kinetic energy term in his Hamiltonian. Weinstein used a similar argument in his setting, the Riemannian metric being replaced by a Finsler metric associated with the more general  $K$ . Due to the fact that the metric degenerates on  $\partial D$ , an approximation argument and a priori bounds which keep the approximate period away from 0 and  $\infty$  are required in both cases.

Weinstein goes on in [21] to prove a result for general Hamiltonian systems:

Theorem 2.3: Suppose  $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$  and  $H^{-1}(1)$  is a manifold which bounds a compact convex region in  $\mathbb{R}^{2n}$ . Then  $(HS)$  possesses a periodic solution on  $H^{-1}(1)$ .

The proof of Theorem 2.3 involves a clever application of Theorem 2.2. A new Hamiltonian on  $\mathbb{R}^{4n}$  is constructed which satisfies the hypotheses of Theorem 2.2 (with  $n$  replaced by  $2n$ ) and for which solutions of the type given in Theorem 2.2 correspond to periodic solutions of (HS) on  $H^{-1}(1)$ .

Simultaneous to Weinstein's work on Theorem 2.3, this author also was studying (HS), but from a totally different point of view, and obtained the following somewhat more general result [22]:

Theorem 2.4: Suppose  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  and  $H^{-1}(1)$  is a manifold which bounds a compact star-shaped region in  $\mathbb{R}^{2n}$ , i.e. there is a  $\xi \in \mathbb{R}^{2n}$  such that, with  $\xi$  as origin,  $H^{-1}(1)$  is radially diffeomorphic to  $S^{2n-1}$ . Then (HS) possesses a periodic solution on  $H^{-1}(1)$ .

To describe the approach taken to (HS) in [22], observe first that the period of any periodic solution on  $H^{-1}(1)$  is not known a priori. It is convenient to rescale the time variable and replace (HS) by

$$(2.5) \quad \dot{z} = \lambda \mathcal{J} H_z(z)$$

where we now seek a  $2\pi$  periodic function  $z$  and a nonzero scalar  $\lambda$  (essentially the unknown period) such that the pair satisfy (2.5). The idea now is to use the calculus of variations to find a solution of (2.5). Let  $(\cdot, \cdot)_{\mathbb{R}^j}$  denote the inner product of two vectors in  $\mathbb{R}^j$ . Formally a critical point  $z$  of the action integral

$$(2.6) \quad A(z) = \int_0^{2\pi} (p, \dot{q})_{\mathbb{R}^n} dt$$

subject to the constraint

$$(2.7) \quad \frac{1}{2\pi} \int_0^{2\pi} H(z) dt = 1$$

has (2.5) as its Euler equation,  $\lambda$  appearing as a Lagrange multiplier due to (2.7). Moreover since  $z$  satisfies (2.5),  $H(z(t)) \equiv c$ , a constant, and by (2.7),  $c = 1$ .

It seems to be a difficult matter to make these heuristics precise in a direct fashion. Instead a finite dimensional approximation argument, looking for solutions in the class of trigonometric polynomials was used in [22]. Observe that  $A(z)$  and the constraint are invariant if  $z(t)$  is replaced by  $z(t+\theta)$  for  $\theta \in [0, 2\pi]$ , i.e. the problem possesses an  $S^1$  symmetry. Thus employing an index theory for such  $S^1$  actions [10] and minimax arguments, critical points can be obtained for an approximating finite dimensional problem. Appropriate bounds for approximate solutions and their associated Lagrange multipliers allow one to pass to a limit and solve (2.5).

An interesting geometrical result concerning the relation between the period and  $H^{-1}(1)$  that comes up in the course of the proof of Theorem 2.4 is the following:

**Theorem 2.8:** Suppose  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  is homogeneous of degree two and positive for  $z \neq 0$ . Let  $r$  and  $R$  denote respectively the radii of the smallest inscribed and largest circumscribed sphere for  $H^{-1}([0, 1])$ . Then (HS) has a  $T$  periodic solution on  $H^{-1}(1)$  with  $\frac{1}{2} r^2 \leq T \leq \frac{1}{2} R^2$ .

The bounds obtained in Theorem 2.8 play a role in the proof of Theorem 2.4. We depart from our chronological development for a moment to

mention the following result which was motivated by Weinstein's Theorem 2.2 :

Theorem 2.9 [23] : Suppose  $H(p, q) = K(p, q) + V(q)$  where  $K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ ,  $V \in C^1(\mathbb{R}^n, \mathbb{R})$ ,  $V$  satisfies  $(V_1) - (V_2)$ , and  $K$  satisfies  $(K_2)$  and

$$(K_3) \quad (p, K_p(p, q))_{\mathbb{R}^n} > 0 \text{ for } p \neq 0 .$$

Then (HS) possesses a periodic solution on  $H^{-1}(1)$  .

Thus Theorem 2.9 replaces the convexity hypothesis  $(K_1)$  by the "starshaped" assumption  $(K_3)$  . The proof of Theorem 2.9 is based on that of Theorem 2.4, the bounds required here being somewhat more difficult to obtain. Theorems 2.1, 2.2, 2.4, and 2.9 all give sufficient geometrical conditions under which (HS) possesses periodic orbits on  $H^{-1}(1)$  . Just how general an energy surface one can take and still be guaranteed the existence of periodic orbits of (HS) on it remains an open question. See e.g. [24] for some conjectures in this direction.

Our discussion up to this point has only dealt with global results for (HS) when the energy is prescribed. In [22] a study also was begun of the existence of periodic solutions of (HS) when the period is prescribed. The simplest such case treated in [22] is:

Theorem 2.10: Suppose  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  and satisfies

$$(H_1) \quad H(z) \geq 0$$

$$(H_2) \quad H(z) = o(|z|^2) \text{ at } z = 0$$

(H<sub>3</sub>) There are constants  $r > 0$  and  $\mu > 2$  such that for all  $|z| > r$ ,

$$0 < \mu H(z) \leq (z, H_z(z))_{\mathbb{R}^{2n}}.$$

Then for any  $T > 0$ , (HS) has a nonconstant  $T$  periodic solution.

A few remarks about this theorem are in order. First integrating the inequality in (H<sub>3</sub>) shows there are constants  $a_1, a_2 > 0$  such that

$$(2.11) \quad H(z) \geq a_1 |z|^\mu - a_2$$

for all  $z \in \mathbb{R}^{2n}$ , i.e.  $H$  grows at a "superquadratic" rate as  $|z| \rightarrow \infty$ . Likewise (H<sub>2</sub>) implies  $H(z) \rightarrow 0$  as  $|z| \rightarrow 0$  at a superquadratic rate.

The proof of Theorem 2.10 is in the same spirit as that of Theorem 2.4. Suppose for convenience we seek a  $2\pi$  periodic solution of (HS). Then any critical point of

$$(2.12) \quad I(z) = A(z) - \int_0^{2\pi} H(z) dt$$

in the class of  $2\pi$  periodic functions is a solution of (HS). To obtain a critical point of  $I$ , one proceeds as in Theorem 2.4 with three main differences: (i) no constraint is involved here; (ii) a minimax argument is given based on (H<sub>1</sub>) - (H<sub>3</sub>) and which avoids the use of symmetries and the index theory of [10]; (iii) an additional difficulty is encountered here due to the presence of the trivial solution  $z \equiv 0$ . To overcome (iii), a comparison argument is employed which shows  $I(z) > 0$  for the solution constructed. Hence (H<sub>1</sub>) and (2.12) imply  $z$  is nonconstant.

In research subsequent to [22], Benci and the author obtained a critical point theorem for indefinite functionals [25] which can be used to



bypass the finite dimensional approximation arguments of [22] and get a critical point of (2.12) directly in the Sobolev space  $(W^{\frac{1}{2},2}(S^1))^{2n}$ .

See also Ekeland [26] who gave a direct proof of a special case of Theorem 2.10.

Several variants of Theorem 2.10 were also proved in [22] including one for (FHS). We will return to this result later when we discuss subharmonics for (FHS). Also discussed in [22] were some results for second order Hamiltonian systems:

$$(2.13) \quad \ddot{q} + V_q(t, q) = 0,$$

where  $q \in \mathbb{R}^n$ , for forced or free  $V$  which satisfy hypotheses like  $(H_1) - (H_3)$ .

Although in its setting Theorem 2.10 guarantees a nonconstant solution of period  $T$  of (HS) for all  $T > 0$ , nothing is implied concerning the existence of a solution having minimal period  $T$ . We suspect that  $(H_1) - (H_3)$  are sufficient to give solutions of (HS) of minimal period  $T$  for any  $T > 0$ . However if  $(H_1) - (H_2)$  are dropped, one cannot expect this to be the case. Indeed suppose  $n = 1$  and consider  $H(z) = g(|z|^2)$  where  $g \in C^\infty(\mathbb{R}, \mathbb{R})$ . Setting  $\zeta = p + iq$ , the corresponding Hamiltonian system can be written in complex form as

$$(2.14) \quad \dot{\zeta} = 2ig'(|\zeta|^2)\zeta.$$

Thus  $\zeta(t) = \zeta_0 \exp[2ig'(|\zeta|^2)t]$  so if  $T$  is the minimal period of  $\zeta(t)$ ,  $T \leq \pi[g'(|\zeta|^2)]^{-1}$ . Consequently  $g' \geq 1$  implies  $T \leq 2\pi$ .

On the other hand if one is not interested in solutions having minimal periods, one can do much better than Theorem 2.10, namely:

**Theorem 2.15** [27] : Suppose  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  and satisfies  $(H_3)$ . Then for all  $T, R > 0$ ,  $(HS)$  has a solution  $z$  of period  $T$  and satisfying  $\|z\|_{L^\infty} > R$ .

Thus one can obtain arbitrarily large  $T$  periodic solutions (which as the above example shows may not have minimal period  $T$ ). The proof of Theorem 2.15 is more complicated than that of Theorem 2.10, the structure given by  $(H_1) - (H_2)$  being replaced by the  $S^1$  invariance of  $I(z)$  as was employed in the proof of Theorem 2.4.

One final remark about the proof of Theorem 2.10: Although it appears to be rather different from Theorem 2.4, the two are closely related and in fact Theorem 2.10 can be used to give a short elementary proof of Theorem 2.4 [27].

At this point in the development of the theory, the convex analysts make their appearance. F. Clarke [28, 29] gave a new and simpler proof of Theorem 2.3. In addition he weakened the smoothness assumptions on  $H$  to merely convexity and continuity so  $(HS)$  becomes an inclusion rather than an equation. However in our description here, we prefer to stay in the classical framework. Part of Clarke's idea is to employ a Legendre transformation to convert the problem to a simpler one. Unlike the usual situation in mechanics, he uses a Legendre transformation in all variables. Equation  $(HS)$  can be written as

$$(2.16) \quad -\dot{p}z = H_z(z)$$

with, in the setting of Theorem 2.3,  $H$  globally convex via a trick of [21] or [22]. Thus  $H_z$  is monotone. In essence, Clarke inverts  $H_z$  in (2.16) transforming it to

$$(2.17) \quad z = H_z^{-1}(-\dot{\mathcal{J}}z) .$$

This new equation in which  $\dot{z}$  is taken to be the independent variable can be given a variational formulation for which a solution can be obtained as a minimum of the corresponding functional.

As another consequence of these ideas, Clarke and Ekeland [30] studied a situation complementary to that of Theorem 2.10 in which  $H$  is "subquadratic" at 0 and  $\infty$ , i.e.

$$(H_4) \quad H(z)|z|^{-2} \rightarrow 0 \quad \text{as} \quad |z| \rightarrow \infty$$

and

$$(H_5) \quad H(z)|z|^{-2} \rightarrow \infty \quad \text{as} \quad |z| \rightarrow 0 .$$

They proved

Theorem 2.18: Suppose  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ ,  $H$  is convex with  $H(0) = 0$ ,  $H_z(0) = 0$ , and satisfies  $(H_4) - (H_5)$ . Then for all  $T > 0$ ,  $(HS)$  has a solution having minimal period  $T$ .

The minimality of  $T$  is a consequence of the characterization of the solution as a minimum of a variational problem. This theorem is the only result we know of in the context of general Hamiltonian systems which obtains information on minimal periods. See also Berger [20] or [31] for results on second order Hamiltonian systems.

In a further application of Legendre transformation ideas in conjunction with minimax arguments and the index theory of [10], Ekeland and Lasry proved a nice result which furnishes a partial globalization of Weinstein's bifurcation theorem (Theorem 1.2). Let  $B_\rho$  denote a Euclidean ball of radius  $\rho$ .

Theorem 2.19: Suppose  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  and  $H^{-1}(1)$  is a manifold which bounds a compact convex domain  $\Omega$ . If there are positive numbers  $r$  and  $R < \sqrt{2}r$  such that  $B_r \subset \Omega \subset B_R$ , then  $H^{-1}(1)$  contains at least  $n$  geometrically distinct periodic solutions of (HS).

Whether or not the restriction on the shape of  $\Omega$  is essential remains an open question. Likewise nothing is known about the number of periodic solutions of (HS) in the settings of Theorems 2.1, 2.2, and 2.9. Also of interest are the number of solutions of (HS) having a given minimal period. No progress seems to have been made in this direction.

In addition to Theorem 2.18, there have been a considerable number of results obtained for subquadratic Hamiltonian systems, both autonomous and forced, satisfying variants of  $(H_4)$  and  $(H_5)$ . We will not go into detail here but they include works by Benci [32-34], Benci and Rabinowitz [25], Brezis and Coron [35], and Coron [36]. Also Amann [37] and Amann and Zehnder [38-39] have studied problems which lie on the border between sub- and superquadratic, namely Hamiltonian systems which are quadratic near 0 and  $\infty$  but have different signatures. Using a global Lyapunov-Schmidt reduction, minimax arguments and index theories, they obtain many existence and multiplicity results for (HS) and (FHS). Some earlier special cases were obtained by D. Clarke [40].

Next we shall describe a contribution to subharmonic solutions of (FHS). As was mentioned earlier, a variant of Theorem 2.10 for (FHS) was given in [22]. It turns out that under the same hypotheses much more is true:

Theorem 2.20 [41]: Suppose  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  and satisfies

(H<sub>6</sub>) There is a constant  $T > 0$  such that  $H(t+T, z) = H(t, z)$  for all  $t \in \mathbb{R}$ ,  $z \in \mathbb{R}^{2n}$ ,

(H<sub>7</sub>) There are constants  $\alpha, \beta > 0$  such that for  $|z| > \beta$   
 $|H_z(t, z)| \leq \alpha(z, H_z(t, z))_{\mathbb{R}^{2n}}$ ,

and (H<sub>1</sub>) - (H<sub>3</sub>) with respect to  $z$ . Then for each  $k \in \mathbb{N}$ , (FHS) possesses a  $kT$  periodic solution  $z_k(t)$ . Moreover infinitely many of the functions  $z_k$  are distinct.

The proof of this result follows from the abstract critical point theorem of [25] combined with some bounds for the critical points and a simple indirect argument showing infinitely many must be distinct. Theorem 2.20 can be viewed as a global relative of the Birkhoff-Lewis theorem where (H<sub>3</sub>) plays the role of the condition on the quartic part of  $H$  in Theorem 1.7. In fact one can prove a local variation on Theorem 1.7 using (H<sub>3</sub>) and Theorem 2.20 as a tool. Clarke and Ekeland [42] have also obtained a result on subharmonics for second order forced Hamiltonian systems (2.13) with convex subquadratic  $V(q)$ . See also [41] for another subquadratic case.

Our final result is a recent theorem of Gluck and Ziller [43] concerning the fixed energy case of (HS) which extends Theorem 2.2.

Theorem 2.21: Suppose  $H(p, q) = K(p, q) + V(q)$  where  
 $K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ ,  $V \in C^2(\mathbb{R}^n, \mathbb{R})$ ,  $K$  satisfies  $(K_1) - (K_2)$ , and  $V$   
 satisfies

$(V_3)$   $\emptyset \neq \{q \in \mathbb{R}^n \mid V(q) \leq 1\}$  is compact and nonempty

and  $(V_2)$ . Then  $(HS)$  possesses a periodic solution on  $H^{-1}(1)$ .

The proof of Theorem 2.21 follows the geometrical approach of [19]  
 and [21] together with some further topological ideas.

*Continued* → In conclusion it should be mentioned that one of the main sources of  
 inspiration for the development of Hamiltonian mechanics was the field of  
 celestial mechanics. In this field, unlike the situations described above,  
 one encounters Hamiltonians which possess singularities. We believe  
 celestial mechanics is a very interesting and possibly fertile proving ground  
 for the further development of the ideas and methods described in this survey.

## References

- [1] Lyapunov, A., Problème générale de la stabilité du mouvement, Ann. Fac. Sci. Toulouse (2), (1907), 203-474.
- [2] Gordon, W. B., A theorem on the existence of periodic solutions to Hamiltonian systems with convex potential, J. Diff. Eq., 10, (1971), 324-335.
- [3] Weinstein, A., Normal modes for non-linear Hamiltonian systems, Inv. Math., 20, (1973), 47-57.
- [4] Moser, J., Periodic orbits near an equilibrium and a theorem by Alan Weinstein, Comm. Pure Appl. Math., 29, (1976), 727-747.
- [5] Bottkol, M., Bifurcation of periodic orbits on manifolds and Hamiltonian systems, thesis, New York University, 1977.
- [6] Weinstein, A., Bifurcations and Hamilton's Principle, Math. Z., 159, (1978), 235-248.
- [7] Böhme, R., Die Lösung der Verzweigungsgleichungen für nichtlineare Eigenvektproblem, Math. Z., 127, (1972), 105-126.
- [8] Marino, A., La biforcazione nel caso variazionale, Conf. Sem. Mat. dell' Univ. Barie, 132, (1977).
- [9] Fadell, E. R. and P. H. Rabinowitz, Bifurcation for odd potential operators and an alternative topological index, J. Funct. Anal., 26, (1977), 48-67.
- [10] Fadell, E. R. and P. H. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems, Inv. Math., 45, (1978), 139-174.

- [11] Chow, S. N. and J. Mallet-Paret, Periodic solutions near an equilibrium of a nonpositive definite Hamiltonian system, Michigan State Univ. preprint.
- [12] Birkhoff, G. D., Une generalization à  $n$ -dimensions du dernier théorème de géométrie de Poincaré, *Compt. Rend. Acad. Sci.*, 192, (1931), 196-198.
- [13] Birkhoff, G. D. and D. C. Lewis, On the periodic motions near a given periodic motion of a dynamical system, *Ann. Mat. Pura Appl.*, 12, (1933), 117-133.
- [14] Lewis, D. C., Sulle oscillazioni periodiche d'une sistema dinamico, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, 19, (1934), 234-237.
- [15] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1978.
- [16] Siegel, C. L. and J. K. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag, 1971.
- [17] Harris, T. C., Periodic solutions of arbitrary long period in Hamiltonian systems, *J. Diff. Eq.*, 4, (1968), 131-141.
- [18] Moser, J., Proof of a generalized fixed point theorem due to G. D. Birkhoff, *Springer Lec. Notes No. 597*, (1977).
- [19] Seifert, H., Periodische Bewegungen mechanischen Systeme, *Math. Z.*, 51, (1948), 197-216.
- [20] Berger, M. S., On a family of periodic solutions of Hamiltonian systems, *J. Diff. Eq.*, 10, (1971), 17-26.
- [21] Weinstein, A., Periodic orbits for convex Hamiltonian systems, *Ann. Math.*, 108, (1978), 507-518.



- [22] Rabinowitz, P. H., Periodic solutions of Hamiltonian systems, Comm. Pure Appl. Math., 31, (1978), 157-184.
- [23] Rabinowitz, P. H., Periodic solutions of a Hamiltonian system on a prescribed energy surface, J. Diff. Eq., 33, (1979), 336-352.
- [24] Weinstein, A., On the hypotheses of Rabinowitz' periodic orbit theorems, J. Diff. Eq., 33, (1979), 353-358.
- [25] Benci, V. and P. H. Rabinowitz, Critical point theorems for indefinite functionals, Inv. math., 52, (1979), 336-352.
- [26] Ekeland, I., Periodic solutions of Hamiltonian equations and a theorem of P. Rabinowitz, to appear J. Diff. Eq.
- [27] Rabinowitz, P. H., A variational method for finding periodic solutions of differential equations, Nonlinear Evolution Equations (M. G. Crandall, editor), Academic Press, (1978), 225-251.
- [28] Clarke, F., Periodic solutions to Hamiltonian inclusions, to appear J. Diff. Eq.
- [29] Clarke, F., A classical variational principle for periodic Hamiltonian trajectories, Proc. Amer. Math. Soc., 76, (1979), 186-188.
- [30] Clarke, F. and I. Ekeland, Hamiltonian trajectories having prescribed minimal period, preprint.
- [31] Berger, M. S., Nonlinearity and Functional Analysis, Academic Press, 1977.
- [32] Benci, V., Some critical point theorems and applications, Comm. Pure Appl. Math., 33, (1980).
- [33] Benci, V., A geometrical index for the group  $S^1$  and some applications to the research of periodic solutions of O.D.E.'s, to appear Comm. Pure Appl. Math.

- [34] Benci, V., On the critical point theory for indefinite functionals in the presence of symmetry, to appear Trans. Amer. Math. Soc.
- [35] Brezis, H. and J. M. Coron, Periodic solutions of nonlinear wave equations and Hamiltonian systems, preprint.
- [36] Coron, J. M., Resolution de l'équation  $Au + Bu = f$  où  $A$  est linéaire autoadjoint et  $B$  déduit d'un potentiel convexe, to appear Ann. Fac. Sci. Toulouse.
- [37] Amann, H., Saddle points and multiple solutions of differential equations, Math. Z., 169, (1979), 127-166.
- [38] Amann, H. and E. Zehnder, Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations, preprint.
- [39] Amann, H. and E. Zehnder, Multiple periodic solutions of asymptotically linear Hamiltonian equations, preprint.
- [40] Clark, D. C., Periodic solutions of variational systems of ordinary differential equations, J. Diff. Eq., 28, (1978), 354-368.
- [41] Rabinowitz, P. H., Subharmonic solutions of Hamiltonian systems, to appear Comm. Pure Appl. Math.
- [42] Clarke, F. H. and I. Ekeland, Nonlinear oscillations and boundary value problems for Hamiltonian systems, preprint.
- [43] Gluck, H. and W. Ziller, Existence of periodic motions of conservative systems, preprint.